TUTORIAL NOTE III

Suppose u is harmonic in Ω. Then integrating by parts we have

$$\int_\Omega u\Delta\varphi = 0 \quad \text{for any } \varphi \in C_c^2(\Omega).$$

The converse is also true and due to Wely.

Exercise 1 (Wely's Lemma). *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$. Suppose $u \in C(\Omega)$ satisfies*

$$\int_\Omega u\Delta\varphi = 0 \quad \text{for any } \varphi \in C_c^2(\Omega). \tag{0.1}$$

Then u is harmonic in Ω.

Proof. First, we claim that for any $B_r(x) \subset \Omega$ there holds

$$r\int_{\partial B_r(x)} u(y)\mathrm{d}S_y = n\int_{B_r(x)} u(y)\mathrm{d}y. \tag{0.2}$$

Indeed, for simplicity we assume that $x = 0$. Set

$$\varphi_k(y,r) = \begin{cases} \left(|y|^2 - r^2\right)^k, & |y| \leq r, \\ 0, & |y| > r, \end{cases}$$

and

$$\psi_k(y,r) = \left(|y|^2 - r^2\right)^{k-2}\left(2(k-1)|y|^2 + n(|y|^2 - r^2)\right)$$

for $k \geq 2$. By an elementary computation, we have $\varphi_k(\cdot, r) \in C_0^2(\Omega)$ and

$$\Delta_y\varphi_k(y,r) = \begin{cases} 2k\psi_k(y,r), & |y| \leq r, \\ 0, & |y| > r. \end{cases}$$

for $k \geq 3$. For $|y| < r$ and $k \geq 3$, we have

$$\begin{aligned} 2k\partial_r\psi_k &= \partial_r\Delta_y\varphi_k \\ &= \Delta_y\partial_r\varphi_k \\ &= -2kr\Delta_y\varphi_{k-1} \\ &= -4k(k-1)r\psi_{k-1}. \end{aligned} \tag{0.3}$$

From (0.1), we see that

$$6\int_{B_r(0)} u(y)\psi_3(y,r)\mathrm{d}y = \int_{B_r(0)} u(y)\Delta_y\varphi_3(y,r)\mathrm{d}y = 0.$$

Now we prove

$$\int_{B_r(0)} u(y)\psi_3(y,r)\mathrm{d}y = 0 \Rightarrow \int_{B_r(0)} u(y)\psi_2(y,r)\mathrm{d}y = 0. \tag{0.4}$$

In fact, we differentiate the first identity in (0.4) with respect to r and get

$$\int_{\partial B_r(0)} u(y)\psi_3(y,r)\mathrm{d}y + \int_{B_r(0)} u(y)\partial_r\psi_3(y,r)\mathrm{d}y = 0.$$

Note that $\psi_3(y,r) = 0$ for $|y| = r$. Therefore, we have

$$\int_{B_r(0)} u(y)\partial_r\psi_3(y,r)\mathrm{d}y = 0.$$

By (0.3), we see that (0.4) is true. We have

$$\int_{B_r(0)} u(y)\left((n+2)|y|^2 - nr^2\right) \mathrm{d}y = 0.$$

Differentiating with respect to r again, we obtain (0.2). Note that, from (0.2), we know that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)\mathrm{d}S_y\right)\\
&= \frac{n}{\omega_n}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^n}\int_{B_r(x)} u(y)\mathrm{d}y\right)\\
&= \frac{n}{\omega_n r^{n+1}}\left(r\int_{\partial B_r(x)} u(y)\mathrm{d}S_y - n\int_{B_r(x)} u(y)\mathrm{d}y\right) = 0.
\end{aligned}$$

This implies

$$u(x) = \lim_{\rho\to 0}\frac{1}{\omega_n \rho^{n-1}}\int_{\partial B_\rho(x)} u(y)\mathrm{d}Sy = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)\mathrm{d}Sy,$$

which means that u is harmonic. □

Second, we introduce the removable singularity for Harmonic function.

Exercise 2. *Suppose u is harmonic in $B_R(0)\setminus\{0\}$ and satisfies*

$$u(x) = \begin{cases} o(\log|x|), & n=2,\\ o(|x|^{2-n}), & n\geq 3, \end{cases} \quad \textit{as } |x|\to 0.$$

Then u can be defined at 0 so that it is C^2 and harmonic in $B_R(0)$.

Proof. We prove the case $n=3$ as an example. Assume u is continuous in $0<|x|\leq R$. Let v solve

$$\begin{cases} \Delta v = 0 & \text{in } B_R,\\ v = u & \text{on } \partial B_R. \end{cases}$$

We claim that $u=v$ in $B_R\setminus\{0\}$. Indeed, we can consider $w=v-u$ in $B_R(0)\setminus\{0\}$ and $M_r = \max\limits_{\partial B_r(0)}|w|$. We observe that

$$|w(x)| \leq M_r r|x|^{-1} \quad \text{on } \partial B_r.$$

Note that w and $|x|^{-1}$ are harmonic in $B_R\setminus B_r$. Hence the maximum principle implies

$$|w(x)| \leq M_r r|x|^{-1} \quad \text{for any } x\in B_R\setminus B_r.$$

Note also that, for all $r\in(0,R)$,

$$M_r \leq \max_{\partial B_r(0)}|u(x)| + \max_{\partial B_r(0)}|v(x)| \leq \max_{\partial B_r(0)}|u(x)| + \max_{\partial B_R(0)}|u(x)|.$$

Combining the above estimates, we have for each fixed $x\neq 0$,

$$|w(x)| \leq |x|^{-1} r\left(\max_{\partial B_r(0)}|u(x)| + \max_{\partial B_R(0)}|u(x)|\right) \to 0 \text{ as } r\to 0,$$

that is $w=0$ in $B_R(0)\setminus\{0\}$. □